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Numerical methods based on rational variable substitution for Wiener–Hopf equations of the second kind ☆

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ABSTRACT

This paper considers numerical methods for Wiener–Hopf equations of the second kind:

$$y(t) + \int_0^\infty k(t-s)y(s)\mathrm{d}s = g(t), \qquad 0 \le t < \infty.$$

By applying rational variable substitution to integrals on the semi-infinite interval $[0, \infty)$ and using the well-known Clenshaw–Curtis quadrature to the resulted integral, we get a Clenshaw–Curtis-Rational (CCR) quadrature rule. We then apply the CCR quadrature to Wiener–Hopf equations. The reduction of singularities in the transformed equation is considered. Numerical examples are given to illustrate the efficiency of the numerical methods proposed in this paper.

1. Introduction

The Wiener–Hopf equation of the second kind is of the form

$$y(t) + \int_0^\infty k(t-s)y(s)\mathrm{d}s = g(t), \quad 0 \le t < \infty, \tag{1.1}$$

where $k(t) \in L_1(\mathbb{R})$ and $g(t) \in L_p[0, \infty)$ $(1 \le p < \infty)$ are given functions. The Wiener–Hopf equation arises in various applications in mathematics and engineering, see for instance [1, pp. 145–146 and pp. 186–189].

Numerical methods for such kinds of integral equations have attracted a lot of attention. Many authors considered truncating the half-line integral equation to obtain a finite-section Wiener–Hopf equation

$$y_R(t) + \int_0^R k(t-s)y_R(s)\mathrm{d}s = g(t), \quad 0 \le t \le R, \tag{1.2}$$

see for instance [2–10]. It has been proved that

$$\lim_{R\to\infty} \|y_R - y\|_{L_p[0,R]} = 0, \quad 1 \le p < \infty,$$

for $g(t) \in L_p[0, \infty)$, see [4]. In the 1990s, the preconditioned conjugate gradient method with a circulant preconditioner for (1.2) was proposed in [5]. Recall that circulant integral operators are integral operators defined by periodic kernel functions.

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Later on, Lin et al. [8] proposed using a convolution operator instead of a circulant operator as a preconditioner. In the 2000s, Kang et al. proposed a Nyström–Clenshaw–Curtis quadrature (a highly accurate numerical scheme based on the Clenshaw–Curtis quadrature) for finite-section Wiener–Hopf integral equations [6,7].

Numerical methods for the half-line equation without truncation have been studied by a number of researchers, see for instance [11–13]. In [11], the authors applied Nyström methods based on composite quadratures to the Wiener–Hopf equation (1.1). Under the assumption that the exact solution decays exponentially, they proved that by choosing the quadrature points properly, the numerical solution converges to the exact solution. Under the same assumption, Mastroianni and Monegato [13] developed a numerical solution method based on a product quadrature which uses the zeros of Laguerre polynomials. In [12], Graham and Mendes considered the case where the kernel function has logarithmic singularity at $s = 0$ and decays exponentially as $s \to \infty$.

In Section 2, we introduce a Clenshaw–Curtis–Rational (CCR) quadrature rule by combining the Clenshaw–Curtis quadrature with the rational variable substitution $s = \frac{\alpha(1-z)}{z+1}$ ($\alpha > 0$ is a parameter):

$$\int_0^\infty f(s)ds \approx Q_n^{(ccr)}(f) = \sum_{j=1}^{n} \frac{2\alpha w_j}{(z_j+1)^2} f\left(\frac{\alpha(1-z_j)}{z_j+1}\right),$$

where z_j and w_j, $j = 1, 2, \ldots, n$, are the quadrature points and the quadrature weights of the n-point Clenshaw–Curtis quadrature (to be specified in Section 2). We then analyze the accuracy of the CCR quadrature.

In Section 3, we discuss numerical solution methods for (1.1). We transform (1.1) into an integral equation of finite interval by substituting the variables t and s by $\frac{\alpha(1-\tau)}{\tau+1}$ and $\frac{\alpha(1-z)}{z+1}$ respectively:

$$Y(\tau) + 2\alpha \int_{-1}^{1} \frac{K(\tau, z)}{(z+1)^2} Y(z)\mathrm{d}z = G(\tau), \quad -1 < \tau \le 1.$$

We will show that $K(\tau, z)$ has singularities along $\tau = z$ when τ tends to -1, see Section 3. To reduce the singularities, we introduce a new function $X(z) \triangleq Y(z)/(z+1)^2$ and reformulate the above integral equation as

$$\left((\tau+1)^2 + 2\alpha \int_{-1}^{1} K(\tau, z)dz\right) X(\tau) + 2\alpha \int_{-1}^{1} K(\tau, z)(X(z) - X(\tau))\mathrm{d}z = G(\tau).$$

We then apply the Clenshaw–Curtis quadrature to $\int_{-1}^{1} \frac{K(\tau,z)}{(z+1)^2} Y(z)\mathrm{d}z$ and $\int_{-1}^{1} K(\tau, z)(X(z) - X(\tau))\mathrm{d}z$ to obtain discretization linear systems corresponding to the above integral equations.

The outline of this paper is as follows. We derive a Clenshaw–Curtis–Rational quadrature rule and analyze its accuracy in Section 2. In Section 3, we apply the CCR quadrature to Wiener–Hopf equations, and discuss the singularities in the equation and ways to reduce them. Numerical examples are given in Section 4 to illustrate the efficiency of the numerical methods proposed in Section 3. Finally, concluding remarks are given in Section 5.

2. Clenshaw–Curtis–Rational quadrature rule

In this section, we derive the Clenshaw–Curtis quadrature rule and a Clenshaw–Curtis–Rational (CCR) quadrature rule, and analyze the accuracy of the two quadratures.

2.1. The Clenshaw–Curtis quadrature rule

Clenshaw–Curtis quadrature rules [14] are closely related to Chebyshev polynomials. Let

$$T_0(z) = 1, \qquad T_i(z) = \cos(i \arccos z), \quad i = 1, 2, \ldots,$$

be the sequence of Chebyshev polynomials. The quadrature rule with quadrature points

$$z_k = \cos((2k-1)\pi/(2n)), \quad k = 1, 2, \ldots, n, \tag{2.1}$$

the roots of $T_n(z)$, is called the classical Clenshaw–Curtis quadrature, and the one with quadrature points $z_i = \cos(i\pi/n)$, $i = 0, 1, \ldots, n$, i.e., the roots of the polynomial $[T_{n+1}(z) - T_{n-1}(z)]$, is called the practical Clenshaw–Curtis quadrature. In the following, we only introduce the classical Clenshaw–Curtis quadrature. For the practical one, please refer to [15].

There are many ways to derive the Clenshaw–Curtis quadrature rule. Here we introduce the quadrature in a way similar to the one presented in [16].

Let $f(z) \in L^2[-1, 1]$, we have the Chebyshev–Fourier expansion:

$$f(z) = \frac{b_0}{2} + \sum_{i=1}^{\infty} b_i T_i(z), \tag{2.2}$$

where the coefficients b_i are given by

$$b_i = \frac{2}{\pi}\int_{-1}^{1}\frac{f(z)T_i(z)}{\sqrt{1-z^2}}\mathrm{d}z, \quad i = 0, 1, 2, \dots. \tag{2.3}$$

From (2.2) we get

$$\int_{-1}^{1} f(z)\mathrm{d}z = b_0 + \sum_{i=1}^{\infty} b_i \int_{-1}^{1} T_i(z)\mathrm{d}z = b_0 + \sum_{i=1}^{\infty}\frac{2b_{2i}}{1-4i^2}. \tag{2.4}$$

Applying the n-point Gauss–Chebyshev quadrature to (2.3), we get

$$b_i \approx \tilde{b}_i = \frac{2}{n}\sum_{k=1}^{n} f(z_k)T_i(z_k) = \frac{2}{n}\sum_{k=1}^{n} f(z_k)\cos\left(\frac{i(2k-1)\pi}{2n}\right), \quad i = 0, 1, 2, \dots \tag{2.5}$$

Recall that the n-point Gauss–Chebyshev quadrature is given by

$$\int_{-1}^{1}\frac{g(z)}{\sqrt{1-z^2}}\mathrm{d}z \approx \frac{\pi}{n}\sum_{k=1}^{n} g(z_k), \tag{2.6}$$

where z_k, $k = 1, 2, \dots, n$, are given by (2.1). Combining (2.4) with (2.5), we get the Clenshaw–Curtis quadrature rule

$$Q_n(f) = \tilde{b}_0 + \sum_{i=1}^{n-1}\tilde{b}_i\int_{-1}^{1} T_i(z)dz = \sum_{k=1}^{n} w_k f(z_k), \tag{2.7}$$

where

$$w_k = \frac{1}{n}\left(2 + \sum_{i=1}^{\lfloor (n-1)/2\rfloor}\frac{4}{1-4i^2}\cos\left(\frac{i(2k-1)\pi}{n}\right)\right), \quad k = 1, \dots, n, \tag{2.8}$$

with $\lfloor \eta \rfloor$ denoting the largest integer that is not larger than η. Notice that the quadrature weights can be efficiently computed via Fast Fourier Transform (FFT) techniques in $O(n \log n)$ operations.

It has been proved that for a function in $C^r[-1, 1]$, the accuracy of the Clenshaw–Curtis quadrature is $O(n^{-(r-1)})$, see [7, Proposition 3]. In the following, we show that the accuracy of the quadrature can be improved to $O(n^{-r})$.

The following Lemmas 1 and 2 are about the coefficients b_i ($i = 0, 1, \dots$) and the approximate coefficients $\tilde{b}_i$ ($i = 0, 1, \dots, n-1$) given by (2.3) and (2.5), respectively.

Lemma 1 ([7]). *Let $f \in C^r[-1, 1]$ with $r > 1$ and let*

$$f(z) = \frac{b_0}{2} + \sum_{i=1}^{\infty} b_i T_i(z), \quad -1 \le z \le 1.$$

Then there exists a constant $c > 0$ such that

$$|b_i| \le c i^{-r},$$

where $f_n(z) = b_0/2 + \sum_{i=1}^{n-1} b_i T_i(z)$.

Lemma 2. *Let $f(z) = \frac{b_0}{2} + \sum_{i=1}^{\infty} b_i T_i(z)$ and $\tilde{b}_i$ ($i = 0, 1, \dots, n-1$) be given by (2.5). Then*

$$\tilde{b}_i = b_i + \sum_{l=1}^{\infty}(-1)^l(b_{2nl+i} + b_{2nl-i}), \quad i = 0, 1, \dots, n-1. \tag{2.9}$$

Proof. From (2.5) we see that

$$\tilde{b}_0 = \frac{2}{n}\sum_{k=1}^{n}\left(\frac{b_0}{2} + \sum_{j=1}^{\infty} b_j T_j(z_k)\right) = b_0 + \frac{2}{n}\sum_{j=1}^{\infty} b_j \sum_{k=1}^{n} T_j(z_k).$$

Let

$$S_j = \sum_{k=1}^{n} T_j(z_k) = \sum_{k=1}^{n}\cos\left(\frac{j(2k-1)\pi}{2n}\right) = \sum_{k=1}^{n}\cos((2k-1)\theta_j),$$

where $\theta_j = j\pi/(2n)$. Notice that for j satisfying $2n \nmid j$,

$$\sum_{k=1}^{n} \cos\left((2k-1)\theta_j\right) = \frac{\sin(2n\theta_j)}{2\sin\theta_j} = \frac{\sin(j\pi)}{2\sin\theta_j} = 0,$$

it is easily seen that

$$S_j = \begin{cases} 0, & 2n \nmid j, \\ (-1)^l n, & j = 2nl. \end{cases} \tag{2.10}$$

Therefore

$$\tilde{b}_0 = b_0 - 2b_{2n} + 2b_{4n} - \cdots = b_0 + \sum_{l=1}^{\infty} (-1)^l (b_{2nl+0} + b_{2nl-0}).$$

For $i = 1, 2, \ldots, n-1$,

$$\begin{aligned}
\tilde{b}_i &= \frac{2}{n} \sum_{k=1}^{n} \left(\frac{b_0}{2} + \sum_{j=1}^{\infty} b_j T_j(z_k) \right) T_i(z_k) \\
&= \frac{b_0}{n} \sum_{k=1}^{n} T_i(z_k) + \frac{2}{n} \sum_{j=1}^{\infty} b_j \sum_{k=1}^{n} T_j(z_k) T_i(z_k) \\
&= \frac{b_0}{n} \sum_{k=1}^{n} T_i(z_k) + \frac{1}{n} \sum_{j=1}^{\infty} b_j \left(\sum_{k=1}^{n} T_{j-i}(z_k) + \sum_{k=1}^{n} T_{j+i}(z_k) \right) \\
&= \frac{b_0}{n} S_i + \frac{1}{n} \sum_{j=1}^{\infty} b_j S_{j-i} + \frac{1}{n} \sum_{j=1}^{\infty} b_j S_{j+i}.
\end{aligned}$$

From (2.10), it follows that

$$\tilde{b}_i = \sum_{l=0}^{\infty} (-1)^l b_{2nl+i} + \sum_{l=1}^{\infty} (-1)^l b_{2nl-i} = b_i + \sum_{l=1}^{\infty} (-1)^l (b_{2nl+i} + b_{2nl-i}).$$

Thus, we obtain (2.9). □

The following theorem gives the accuracy of the Clenshaw–Curtis quadrature.

Theorem 1. *Let $f \in C^r[-1, 1]$ with $r > 1$ and $Q_n(f)$ be the Clenshaw–Curtis quadrature defined by (2.7). Then*

$$|I(f) - Q_n(f)| = \left| \int_{-1}^{1} f(z)dz - \sum_{j=1}^{n} w_j f(z_j) \right| = O(n^{-r}).$$

Proof. Since $f(z) \in C^r[-1, 1]$, there exists a constant $c > 0$ such that

$$|b_i| \le c i^{-r}.$$

Assume n is even, from (2.4) and (2.7), we get

$$\begin{aligned}
|I(f) - Q_n(f)| &= \left| b_0 + \sum_{i=1}^{\infty} \frac{2b_{2i}}{1-4i^2} - \tilde{b}_0 - \sum_{i=1}^{n/2-1} \frac{2\tilde{b}_{2i}}{1-4i^2} \right| \\
&\le |b_0 - \tilde{b}_0| + \sum_{i=1}^{n/2-1} \frac{2|b_{2i} - \tilde{b}_{2i}|}{4i^2-1} + \sum_{i=n/2}^{\infty} \frac{2|b_{2i}|}{4i^2-1}.
\end{aligned} \tag{2.11}$$

Using (2.9), we get

$$|b_0 - \tilde{b}_0| = \left| 2\sum_{l=1}^{\infty} (-1)^{l-1} b_{2nl} \right| \le 2c \sum_{l=1}^{\infty} (2nl)^{-r} = 2^{-r+1} c n^{-r} \sum_{l=1}^{\infty} l^{-r} = c_1 n^{-r}.$$

Similarly, for $i = 1, 2, \ldots, n/2 - 1$,

$$|b_{2i} - \tilde{b}_{2i}| \le \sum_{l=1}^{\infty} (|b_{2nl+2i}| + |b_{2nl-2i}|) \le c \sum_{l=1}^{\infty} [(2nl + 2i)^{-r} + (2nl - 2i)^{-r}] \le c_2 n^{-r}.$$

Therefore

$$|b_0 - \tilde{b}_0| + \sum_{i=1}^{n/2-1} \frac{2|b_{2i} - \tilde{b}_{2i}|}{4i^2 - 1} \le \max\{c_1, c_2\} n^{-r} \left\{ 1 + \sum_{i=1}^{n/2-1} \frac{2}{(2i-1)(2i+1)} \right\}$$
$$= \max\{c_1, c_2\} n^{-r} \left(2 - \frac{1}{n-1} \right) = O(n^{-r}). \tag{2.12}$$

Moreover,

$$\sum_{i=n/2}^{\infty} \frac{2|b_{2i}|}{4i^2 - 1} \le c \sum_{i=n/2}^{\infty} \frac{2(2i)^{-r}}{4i^2 - 1} < cn^{-r} \sum_{i=n/2}^{\infty} \frac{2}{(2i-1)(2i+1)} = O(n^{-r-1}). \tag{2.13}$$

From (2.11)–(2.13), the theorem follows. □

From the proof of Theorem 1 we see that the accuracy of the Clenshaw–Curtis quadrature is determined by the decay rate of the coefficients in the expansion of $f(z)$. We emphasize that error bound for $Q_n(f)$ is not sharp. For example, let $f(z) = (1+z)^{3/2}$. We have $f(z) \in C^1[-1, 1]$ and $f(z) \notin C^2[-1, 1]$. However, the expansion coefficients of $f(z)$ are given by

$$b_i = \frac{48\sqrt{2}}{\pi(4i^2 - 9)(4i^2 - 1)} = O(i^{-4}).$$

Therefore, when the Clenshaw–Curtis quadrature is used to evaluate the integral $\int_{-1}^{1} (1 + z)^{3/2} dz$, the accuracy is $O(n^{-4})$ rather than $O(n^{-1})$.

2.2. The Clenshaw–Curtis–Rational quadrature rule

To apply the Clenshaw–Curtis quadrature to $\int_0^{\infty} f(s) ds$, we introduce the rational variable substitution $s = \frac{\alpha(1-z)}{z+1}$, i.e., $z = \frac{\alpha(1-s)}{s+\alpha}$, where $\alpha > 0$ is a parameter. Then

$$\int_0^{\infty} f(s) ds = \int_{-1}^{1} f\left(\frac{\alpha(1-z)}{z+1} \right) \frac{2\alpha}{(z+1)^2} dz.$$

If we apply the Gauss–Legendre quadrature to the integral on the right-hand side, we get the Gauss–Rational quadrature [16]. Applying the n-point Clenshaw–Curtis quadrature to the above integral, we get

$$\int_0^{\infty} f(s) ds \approx Q_n^{(ccr)}(f) = \sum_{j=1}^{n} \frac{2\alpha w_j}{(z_j + 1)^2} f\left(\frac{\alpha(1 - z_j)}{z_j + 1} \right), \tag{2.14}$$

where $z_1, \ldots, z_n$ are the roots of $T_n(z)$ given by (2.1) and $w_1, \ldots, w_n$ are given by (2.8). We call the quadrature rule (2.14) a Clenshaw–Curtis–Rational (CCR) quadrature rule.

In the following, we analyze the accuracy of the CCR quadrature. We first prove Lemma 3 that is required in proving the main result Theorem 2.

Lemma 3. *Let $r \ge 2$ be an integer. Then*

$$\frac{1}{2} - \frac{1}{3} \binom{r-2}{1} + \cdots + (-1)^{r-1} \frac{1}{r-1} \binom{r-2}{r-3} + (-1)^r \frac{1}{r} = \frac{1}{r(r-1)}, \tag{2.15}$$

where $\binom{m}{n} = \frac{m!}{n!(m-n)!}$.

Proof. Consider the polynomial

$$p(z) = \frac{z^2}{2} - \frac{z^3}{3} \binom{r-2}{1} + \cdots + (-1)^{r-1} \frac{z^{r-1}}{r-1} \binom{r-2}{r-3} + (-1)^r \frac{z^r}{r}.$$

We only require to prove that $p(1) = \frac{1}{r(r-1)}$. The derivative of $p(z)$ is

$$\begin{aligned} p'(z) &= z - z^2 \binom{r-2}{1} + \cdots + (-1)^{r-1} z^{r-2} \binom{r-2}{r-3} + (-1)^r z^{r-1} \\ &= z \left[1 + (-z)\binom{r-2}{1} + \cdots + (-z)^{r-3} \binom{r-2}{r-3} + (-z)^{r-2} \right] \\ &= z(1-z)^{r-2} = (1-z)^{r-2} - (1-z)^{r-1}. \end{aligned}$$

Thus

$$p(z) = -\frac{(1-z)^{r-1}}{r-1} + \frac{(1-z)^r}{r} + c,$$

where c is a constant. From $p(0) = 0$, it follows that $c = \frac{1}{r(r-1)}$ and $p(1) = c = \frac{1}{r(r-1)}$. $\square$

Theorem 2. *Let $f_1(z) = f((1-z)/(z+1))$, $-1 < z \le 1$. If $f_1(z)$ satisfies the following conditions:*

1. $f_1(z) \in C^r(-1, 1]$ *with* $r > 2$.
2. $\lim_{z\to-1^+} f_1(z) = \lim_{z\to-1^+} f_1'(z) = 0$.
3. *For each $k = 2, 3, \ldots, r$, the limit $\lim_{z\to-1^+} f_1^{(k)}(z)$ exists.*

Then the CCR quadrature (2.14) has the accuracy of order $O(n^{-r+2})$.

Proof. In the light of Theorem 1, we only require to prove that the integrand $f_\alpha(z)/(z+1)^2$ in (2.14) is in the set $C^{r-2}[-1, 1]$, where

$$f_\alpha(z) = f\left(\frac{\alpha(1-z)}{z+1}\right), \quad -1 < z \le 1.$$

Assign $f_1(-1) = 0$, $f_1'(-1) = 0$, and $f_1^{(k)}(-1) = \lim_{z\to-1^+} f_1^{(k)}(z)$, $k = 2, \ldots, r$, then $f_1(z) \in C^r[-1, 1]$. It can be checked that

$$f_\alpha(z) = f_1(u(z)) \quad \text{with } u(z) = \frac{z+1-\alpha(1-z)}{\alpha(1-z)+(z+1)}.$$

Since $\alpha > 0$, the denominator of $u(z)$ is greater than 0 for $z \in [-1, 1]$. More precisely,

$$\min(2, 2\alpha) \le \alpha(1-z) + (z+1) \le \max(2, 2\alpha), \quad z \in [-1, 1].$$

Therefore, $u(z) \in C^\infty[-1, 1]$. It follows that for each $k \in \{0, 1, \ldots, r\}$, the limit $\lim_{z\to-1^+} f_\alpha^{(k)}(z)$ exists. Assign $f_\alpha^{(k)}(-1) = \lim_{z\to-1^+} f_\alpha^{(k)}(z)$, $k = 0, 1, \ldots, r$, then $f_\alpha(z) \in C^r[-1, 1]$. Moreover,

$$\lim_{z\to-1^+} f_\alpha(z) = \lim_{z\to-1^+} f_1(u(z)) = f_1(-1) = 0.$$
$$\lim_{z\to-1^+} f_\alpha'(z) = \lim_{z\to-1^+} u'(z) f_1'(u(z)) = u'(-1) f_1'(-1) = 0.$$

Since $f_\alpha(z) \in C^r[-1, 1]$ and $f_\alpha(-1) = f_\alpha'(-1) = 0$, by using Taylor expansion with remainder, we get

$$\frac{f_\alpha(z)}{(z+1)^2} = \frac{f_\alpha''(-1)}{2!} + \frac{f_\alpha^{(3)}(-1)}{3!}(z+1) + \cdots + \frac{f_\alpha^{(r-1)}(-1)}{(r-1)!}(z+1)^{r-3} + \frac{1}{(r-1)!(z+1)^2} \int_{-1}^{z} f_\alpha^{(r)}(t)(z-t)^{r-1} dt,$$

see for instance [16, p. 47]. The rest is to prove that the function $h(z)$ defined by

$$h(z) = \frac{1}{(z+1)^2} \int_{-1}^{z} f_\alpha^{(r)}(t)(z-t)^{r-1} dt$$

satisfies $h(z) \in C^{r-2}(-1, 1]$ and $h^{(k)}(z)$ has a limit when $z \to -1^+$ for each $k \in \{0, 1, \ldots, r-2\}$. Since $\int_{-1}^{z} f_\alpha^{(r)}(t)(z-t)^{r-1} dt \in C^r(-1, 1]$ and $1/(z+1)^2 \in C^\infty(-1, 1]$, for $k \le r$ and $z \in (-1, 1]$, we have

$$\begin{aligned} h^{(k)}(z) &= \sum_{i=0}^{k} \binom{k}{i} \left(\frac{1}{(z+1)^2}\right)^{(i)} \left(\int_{-1}^{z} f_\alpha^{(r)}(t)(z-t)^{r-1} dt\right)^{(k-i)} \\ &= \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i (i+1)!}{(z+1)^{i+2}} \cdot \prod_{l=1}^{k-i} (r-l) \cdot \int_{-1}^{z} f_\alpha^{(r)}(t)(z-t)^{r-1-k+i} dt \\ &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{(i+1)!(r-1)!}{(r-1-k+i)!} \cdot \frac{\int_{-1}^{z} f_\alpha^{(r)}(t)(z-t)^{r-1-k+i} dt}{(z+1)^{i+2}}. \end{aligned} \tag{2.16}$$

Therefore $h(z) \in C^r(-1, 1]$. To prove that $h^{(k)}(z)$ has a limit when $z \to -1^+$ for each $k \in \{0, 1, \ldots, r-2\}$, we first note that

$$\lim_{z\to -1^+} \frac{\int_{-1}^{z} f_\alpha^{(r)}(t)(z-t)^l dt}{(z+1)^i} = 0, \quad i \le l \le r-1. \tag{2.17}$$

From (2.16) and (2.17) we see that if $k \le r-3$, i.e., $r-k+i-1 \ge 3+i-1 = i+2$, then

$$\lim_{z\to -1^+} h^{(k)}(z) = 0.$$

Finally, note that for $i = 1, 2, \ldots, r$,

$$\begin{aligned}\lim_{z\to -1^+} \frac{\int_{-1}^{z} f_\alpha^{(r)}(t)(z-t)^{i-1} dt}{(z+1)^i} &= \lim_{z\to -1^+} \left[\frac{f_\alpha^{(r)}(-1)}{i} + \frac{\int_{-1}^{z} f_\alpha^{(r)}(t)(z-t)^i dt}{i(z+1)^i}\right] \\ &= \frac{f_\alpha^{(r)}(-1)}{i},\end{aligned}$$

it follows from (2.15) and (2.16) that

$$\lim_{z\to -1^+} h^{(r-2)}(z) = \sum_{i=0}^{r-2} (-1)^i \binom{r-2}{i} (r-1)! \cdot \frac{f_\alpha^{(r)}(-1)}{i+2} = \frac{(r-2)! f_\alpha^{(r)}(-1)}{r}.$$

The proof of the theorem is completed. □

We note that the CCR quadrature can be very accurate for some functions. For example, consider the function $f(s) = 1/(1+s^2)$. We have $f_1(z) = \frac{(z+1)^2}{(z+1)^2+(1-z)^2} \in C^\infty[-1, 1]$. Moreover, $f_1(-1) = 0$ and $f_1'(-1) = \frac{1-z^2}{(z^2+1)^2}|_{z=-1} = 0$. Therefore, the CCR quadrature for $\int_0^\infty 1/(1+s^2)ds$ is highly accurate. Our tests show that when $\alpha = 1$, the error in $Q_{40}^{(ccr)}(f)$ is 2.2204e−016.

3. Application of the CCR quadrature to Wiener–Hopf equations

In this section, we consider numerical solution methods for the Wiener–Hopf equation (1.1):

$$y(t) + \int_0^\infty k(t-s)y(s)ds = g(t), \quad 0 \le t < \infty. \tag{3.1}$$

We assume that $k(t) \in L_1(\mathbb{R})$ is semi-smooth, i.e., $k(t) \in C^r(0, \infty)$ and $k(t) \in C^r(-\infty, 0)$ for certain positive integer r and $y(t) \in C^r[0, \infty)$ satisfying

$$|y(t)| \le \frac{c}{t^2} \tag{3.2}$$

for certain $c > 0$ for large t.

Substituting the variables t and s in (3.1) by $\frac{\alpha(1-\tau)}{\tau+1}$ and $\frac{\alpha(1-z)}{z+1}$ respectively, we get the following integral equation

$$Y(\tau) + 2\alpha \int_{-1}^{1} \frac{K(\tau, z)}{(z+1)^2} Y(z)dz = G(\tau), \quad -1 < \tau \le 1, \tag{3.3}$$

where

$$K(\tau, z) = k\left(\frac{\alpha(1-\tau)}{\tau+1} - \frac{\alpha(1-z)}{z+1}\right), \qquad Y(\tau) = y\left(\frac{\alpha(1-\tau)}{\tau+1}\right), \qquad G(\tau) = g\left(\frac{\alpha(1-\tau)}{\tau+1}\right).$$

Applying the Clenshaw–Curtis quadrature to (3.3) we get the following discretization linear system

$$(I_n + 2\alpha KZW)\mathbf{y}_n = \mathbf{g}_n, \tag{3.4}$$

where I_n is the $n \times n$ identity matrix, $K = [K(z_i, z_j)]_{i,j=1}^n$, $Z = \operatorname{diag}((z_1+1)^{-2}, (z_2+1)^{-2}, \ldots, (z_n+1)^{-2})$, $W = \operatorname{diag}(w_1, \ldots, w_n)$, $\mathbf{y}_n = [y_i]_{i=1}^n$, and $\mathbf{g}_n = [G(z_i)]_{i=1}^n$. Here $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ are the quadrature points and the quadrature weights of the Clenshaw–Curtis quadrature, respectively.

We notice that the kernel function of (3.3)

$$\frac{K(\tau, z)}{(z+1)^2} = \frac{k(2\alpha(z-\tau)/[(\tau+1)(z+1)])}{(z+1)^2}$$

has singularities along $z = \tau$ as τ tends to -1 since the denominators $\tau + 1$, $z + 1$, and $(z + 1)^2$ tend to infinity. On the other hand, under the assumption (3.2), the integrand of (3.3) satisfies

$$\left|\frac{K(\tau,z)}{(z+1)^2}Y(z)\right| = \left|\frac{K(\tau,z)}{(z+1)^2}y\left(\frac{\alpha(1-z)}{z+1}\right)\right| \le \left|\frac{K(\tau,z)}{(z+1)^2}c\left(\frac{\alpha(1-z)}{z+1}\right)^{-2}\right| = \left|\frac{cK(\tau,z)}{\alpha^2(1-z)^2}\right|,$$

i.e., $\frac{K(\tau,z)}{(z+1)^2}Y(z)$ is bounded.

Now we consider a way to reduce the singularities. Since the factor $1/(z+1)^2$ in the kernel function of (3.3) is independent of τ, we define a new function $X(z) \triangleq Y(z)/(z+1)^2$ and then subtract the singularities by reformulating (3.3) as

$$\left((\tau+1)^2 + 2\alpha\int_{-1}^{1}K(\tau,z)dz\right)X(\tau) + 2\alpha\int_{-1}^{1}K(\tau,z)(X(z)-X(\tau))\mathrm{d}z = G(\tau). \tag{3.5}$$

Applying the Clenshaw–Curtis quadrature to $\int_{-1}^{1}K(\tau,z)(X(z)-X(\tau))\mathrm{d}z$, we obtain the discretization equation

$$\left((z_i+1)^2 + 2\alpha\int_{-1}^{1}K(z_i,z)dz - 2\alpha\sum_{j=1}^{n}w_jK(z_i,z_j)\right)x_i + 2\alpha\sum_{j=1}^{n}K(\tau,z_j)w_jx_j = G(z_i), \quad i = 1,2,\ldots,n.$$

Let $\mathbf{x}_n = [x_i]_{i=1}^n$ and D be the diagonal matrix with diagonal entries given by

$$[D]_{ii} = (z_i+1)^2 + 2\alpha\int_{-1}^{1}K(z_i,z)dz - 2\alpha\sum_{j=1}^{n}K(z_i,z_j)w_j, \quad i = 1,2,\ldots,n,$$

we get the matrix form of the above discretization equation

$$(D + 2\alpha KW)\mathbf{x}_n = \mathbf{g}_n. \tag{3.6}$$

After getting $\mathbf{x}_n$, we can another approximate solution $\tilde{\mathbf{y}}_n = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)^T$:

$$\tilde{y}_i = x_i(z_i+1)^2, \quad i = 1,2,\ldots,n. \tag{3.7}$$

To compute the diagonal matrix D, we are required to evaluate the values of $2\alpha\int_{-1}^{1}K(z_i,z)dz$. In our numerical tests in Section 4, we evaluate $2\alpha\int_{-1}^{1}K(z_i,z)dz$ as follows. Let $t_i = \frac{\alpha(1-z_i)}{z_i+1}$, then

$$\begin{aligned}2\alpha\int_{-1}^{1}K(z_i,z)dz &= \int_0^\infty k(t_i-s)\left(\frac{2\alpha}{s+\alpha}\right)^2 ds\\ &= \int_0^{t_i} k(t_i-s)\left(\frac{2\alpha}{s+\alpha}\right)^2 ds + \int_0^\infty k(-s)\left(\frac{2\alpha}{s+t_i+\alpha}\right)^2 ds.\end{aligned}$$

For the first term, we replace s by $\frac{t_i}{2}(z+1)$ and then use the Clenshaw–Curtis quadrature (2.7). For the second term, we use the CCR quadrature (2.14). Thus

$$\begin{aligned}2\alpha\int_{-1}^{1}K(z_i,z)dz &\approx \frac{t_i}{2}\sum_{j=1}^{n}w_jk\left(\frac{t_i}{2}(1-z_j)\right)\left(\frac{2\alpha}{t_i(z_j+1)/2+\alpha}\right)^2 + \sum_{j=1}^{n}\frac{(2\alpha)^3w_j}{[2\alpha+t_i(z_j+1)]^2}k\left(\frac{-\alpha(1-z_j)}{z_j+1}\right)\\ &= \sum_{j=1}^{n}\frac{8\alpha^3w_j}{[2\alpha+t_i(z_j+1)]^2}\left[t_ik\left(\frac{t_i}{2}(1-z_j)\right) + k\left(\frac{-\alpha(1-z_j)}{z_j+1}\right)\right].\end{aligned} \tag{3.8}$$

When the Clenshaw–Curtis quadrature is applied to integral equations (3.3) and (3.5), the accuracy of the numerical solutions depends on the smoothness of $K(\tau,z)Y(z)/(z+1)^2$ and $K(\tau,z)(X(z)-X(\tau))$ respectively. Since the singularities in $K(\tau,z)(X(z)-X(\tau))$ are weaker than those in $K(\tau,z)Y(z)/(z+1)^2$, we expect that the numerical solution $\tilde{\mathbf{y}}_n$ obtained from (3.7) is more accurate than $\mathbf{y}_n$ obtained from (3.3). Numerical experiments presented in Section 4 verify our deduction.

To end this section, we show the singularities in $K(\tau,z)Y(z)/(z+1)^2$ and $K(\tau,z)(X(z)-X(\tau))$ by a simple example. Let $k(t) = 1/(1+t^2)$ and $y(t) = 1/(1+t^2)$, then

$$h_{1,\tau}(z) \equiv K(\tau,z)Y(z)/(z+1)^2 = K(\tau,z)\cdot\frac{1}{(z+1)^2+\alpha^2(1-z)^2}$$

and

$$h_{2,\tau}(z) \equiv K(\tau,z)(X(z)-X(\tau)) = K(\tau,z)\cdot\left(\frac{1}{(z+1)^2+\alpha^2(1-z)^2} - \frac{1}{(\tau+1)^2+\alpha^2(1-\tau)^2}\right)$$

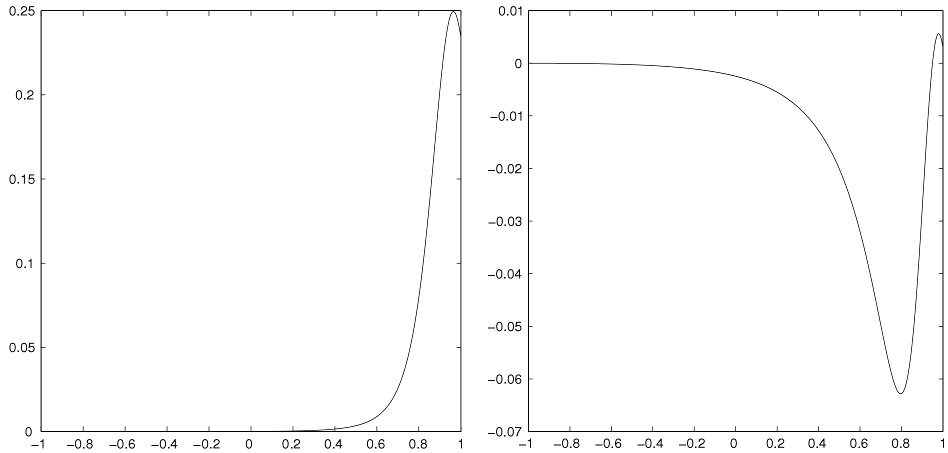

Fig. 1. Curves of the integrands in (3.3) (left) and (3.5) (right): $\tau = 0.95$.

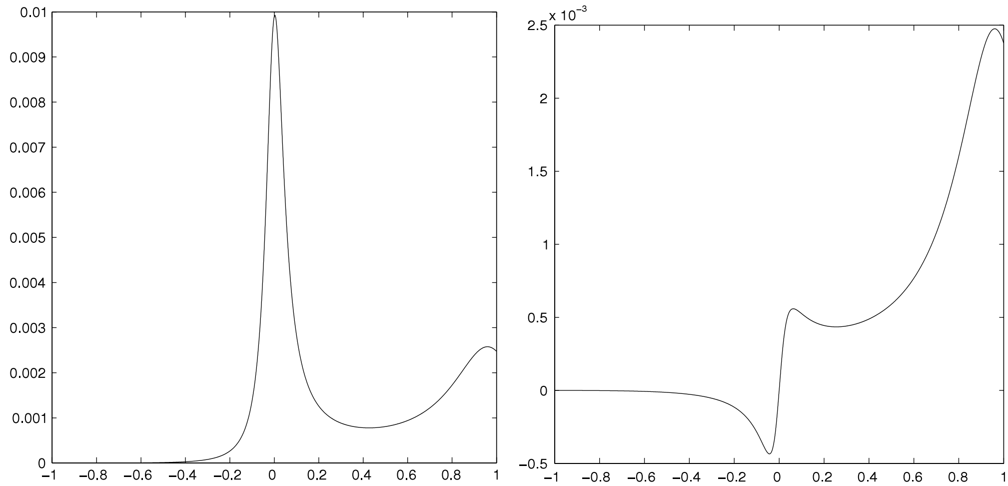

Fig. 2. Curves of the integrands in (3.3) (left) and (3.5) (right): $\tau = 0$.

respectively, where

$$K(\tau, z) = \frac{(\tau+1)^2(z+1)^2}{(\tau+1)^2(z+1)^2 + 4\alpha^2(z-\tau)^2}.$$

We show the figures of the integrands (fixed τ) $h_{1,\tau}(z)$ and $h_{2,\tau}(z)$ in the following for $\alpha = 10$. The curves of $h_{1,\tau}(z)$ and $h_{2,\tau}(z)$ for three τ's: 0.95 (close to 1), 0, and -0.95 (close to -1), are shown in Figs. 1–3, respectively. We can see from Fig. 1 that for $\tau = 0.95$, both integrands $h_{1,\tau}(z)$ and $h_{1,\tau}(z)$ are very smooth. From Fig. 2 we observe that for $\tau = 0$, $h_{1,\tau}(z)$ has a sharp peak while $h_{2,\tau}(z)$ looks quite smooth. Fig. 3 shows the case where $\tau = -0.95$. In this case, both integrands have singularities: $h_{1,\tau}(z)$ has strong singularity at $z = \tau$ and $h_{2,\tau}(z)$ has weak singularity. It must be noted that $h_{2,\tau}(z)$ is very close to zero when τ is close to -1, which guarantees high accuracy of the CCR quadrature when it is applied to discretize the integral equation (3.5).

4. Numerical experiments

We first note that Kang, Koltracht, and Rawitscher have proposed a Nyström–Clenshaw–Curtis (NCC) quadrature for the finite-section Wiener–Hopf equation (1.2) [6,7]. The NCC quadrature is a highly accurate quadrature which is derived by using the Clenshaw–Curtis quadrature cleverly. In this section, we make a brief comparison on accuracy of the numerical solutions obtained by using the CCR quadrature and by using the composite NCC quadrature. In the experiments, we set $\alpha = 10$, cf. (3.3) and (3.5). Since the linear systems (3.4) and (3.6) are small, we use the Gaussian elimination method to solve them.

We use the following notation. The symbols $f_n(t)$ and $\tilde{f}_n(t)$ denote the interpolating polynomials obtained by interpolating the sets of points $\{(t_i, y_i) | i = 1, \ldots, n\}$ and $\{(t_i, \tilde{y}_i) | i = 1, \ldots, n\}$ respectively, where $t_1, \ldots, t_n$ are the quadrature

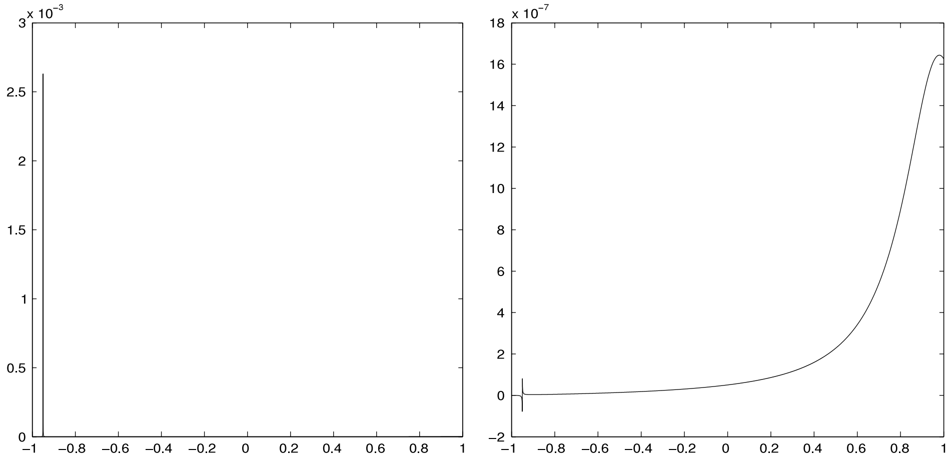

Fig. 3. Curves of the integrands in (3.3) (left) and (3.5) (right): $\tau = -0.95$.

Table 1
The number of quadrature points and errors in the numerical solutions for Example 1.

n	$err(f_n)$	$err(\tilde{f}_n)$	$err(f_{4,n}^{40})$	$err(f_{8,n}^{40})$	$err(f_{4,n}^{80})$	$err(f_{8,n}^{80})$
32	4.862e−5	3.078e−6	0.5351	1.202	1.000	0.9971
64	2.911e−6	4.957e−8	8.609e−4	0.0464	0.3276	0.5351
128	1.808e−7	7.651e−10	2.636e−10	2.859e−7	2.223e−5	8.609e−4
256	1.130e−8	1.192e−11	2.567e−10	2.877e−13	2.148e−5	2.636e−10
512	7.064e−10	1.863e−13	3.308e−10	3.331e−13	1.774e−5	2.567e−10

points, and $\mathbf{y}_n = [y_1, \ldots, y_n]^T$ and $\tilde{\mathbf{y}}_n = [\tilde{y}_1, \ldots, \tilde{y}_n]^T$ are the solutions of (3.4) and (3.7) respectively. Let $\mathbf{f}_{k,n}^R$ denote the numerical solution obtained by using the following composite NCC quadrature: partition the truncation interval $[0, R]$ into k subintervals of the same length and use the (n/k)-point NCC quadrature in each subinterval. The symbol $f_{k,n}^R(t)$ denotes the corresponding piecewise interpolating polynomial. In Tables 1–3, we estimate the error in the numerical solution $\hat{f}$ by

$$err(\hat{f}) = \max\left\{ \max_{i=1,\ldots,1000} |\hat{f}(i/10) - y(i/10)|,\ |y(R)| \right\},$$

where $y(t)$ is the exact solution (for $f_n(t)$ and $\tilde{f}_n(t)$, we set $R = \infty$ and therefore $y(R) = 0$).

Example 1 (*[6, Chapter 3]*)**.** Consider

$$y(t) + \int_0^\infty k(t-s)y(s)\mathrm{d}s = (2 + t + t^2/2 + t^3/3)e^{-t}, \quad 0 \le t < \infty,$$

where $k(t) = \left(1 + |t| + |t|^2\right)e^{-|t|}$. The true solution of the above equation is $y(t) = e^{-t}$.

From Table 1 we observe that with the CCR method, the accuracy of the numerical solution is about $O(n^{-6})$, and $\tilde{f}_n(t)$ is more accurate than $f_n(t)$. For the NCC method, we don't need to choose a large R because the true solution decays very fast. We notice that due to the factor $e^{-|t-s|}$ in the kernel (in the NCC method, it is required that both e^{s-t} and e^{t-s} are bounded), we have to partition the interval $[0, R]$ into subintervals of small length, otherwise, there exist entries in the coefficient matrix that are very large. From Table 1 we see that for most cases, the CCR method performs better than the NCC method.

Example 2 (*[11]*)**.** Consider

$$y(t) - \int_0^\infty \frac{\sqrt{3}}{2\pi}\,\mathrm{sech}(t-s)y(s)\mathrm{d}s = g(t), \quad 0 \le t < \infty,$$

where

$$g(t) = e^{-t/3}\left(\frac{1}{4} + \frac{\sqrt{3}}{2\pi}\ln\frac{u+1}{\sqrt{u^2-u+1}} + \frac{3}{2\pi}\arctan\frac{2u-1}{\sqrt{3}}\right),$$

with $u = e^{-2t/3}$. The true solution is $y(t) = e^{-t/3}$.

Table 2
The number of quadrature points and errors in the numerical solutions for Example 2.

n	$err(f_n)$	$err(\tilde{f}_n)$	$err(f_{1,n}^{40})$	$err(f_{2,n}^{40})$	$err(f_{1,n}^{80})$	$err(f_{2,n}^{80})$
32	1.704e−2	2.743e−4	0.0026	0.0107	0.0648	0.3194
64	3.523e−4	1.032e−5	6.445e−6	4.285e−5	3.603e−4	0.0026
128	6.718e−6	1.385e−7	1.620e−6	1.620e−6	3.596e−7	6.445e−6
256	1.373e−8	4.642e−10	1.620e−6	1.620e−6	2.668e−12	1.441e−10
512	8.253e−12	2.541e−13	1.620e−6	1.620e−6	2.623e−12	2.623e−12

Table 3
The number of quadrature points and errors in the numerical solutions for Example 3.

n	$err(f_n)$	$err(\tilde{f}_n)$	$err(f_{1,n}^{160})$	$err(f_{2,n}^{160})$	$err(f_{1,n}^{640})$	$err(f_{2,n}^{640})$
32	8.172e−4	8.439e−6	0.0897	0.1842	0.1204	0.4631
64	2.512e−4	3.161e−7	0.0013	0.0064	0.0900	0.1862
128	8.646e−5	1.928e−8	3.906e−5	1.298e−4	0.0013	0.0065
256	3.148e−5	1.604e−9	3.906e−5	3.906e−5	3.990e−5	1.647e−4
512	1.186e−5	1.432e−10	3.906e−5	3.906e−5	2.500e−6	7.628e−6

We observe from Table 2 that as n increases, the errors in $f_n(t)$ and $\tilde{f}_n(t)$ decay rather rapidly. In fact, the convergence is superalgebraic. On the other hand, since the exact solution does not decay as fast as the solution in Example 1, a large R should be chosen to avoid large truncation error. Since the kernel function is smooth, we set k to small integers for the composite NCC method. From the numerical results we see that $R = 80$ is a suitable choice. Furthermore, for most of n, $\tilde{f}_n(t)$ is more accurate than $f_{k,n}^R(t)$.

Example 3. Consider

$$y(t) + \int_0^\infty \frac{1}{1+(t-s)^2} y(s)\mathrm{d}s = g(t), \quad 0 \le t < \infty,$$

where

$$g(t) = \frac{1}{1+t^2} + \frac{1}{4+t^2}(\pi + \arctan(t)) + \frac{\ln(1+t^2)}{t(4+t^2)}.$$

The true solution of the equation is $y(t) = 1/(1+t^2)$.

In this example, the exact solution decays quite slowly and the kernel function is smooth. A very large R should be chosen to avoid large truncation error. On the other hand, due to the smoothness of the kernel, we can set the number of subintervals k small. We observe from Table 3 that $\tilde{f}_n(t)$ is much more accurate than $f_n(t)$, and the error in $\tilde{f}_n(t)$ is about $O(n^{-3.5})$. The performance of the NCC method is not good for this example (with 512 quadrature points, the best numerical solution has the accuracy 2.500e−6). We can see from Table 3 that for all n, $\tilde{f}_n(t)$ is much more accurate than $f_{k,n}^R(t)$.

5. Concluding remarks

In this paper, we proposed a Clenshaw–Curtis–Rational quadrature rule which is based on a rational variable substitution and the Clenshaw–Curtis quadrature, and apply it to solve Wiener–Hopf equations of the second kind. Our results indicate that the CCR method is well suited for integrands that decay exponentially as well as for those that decay quadratically.

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